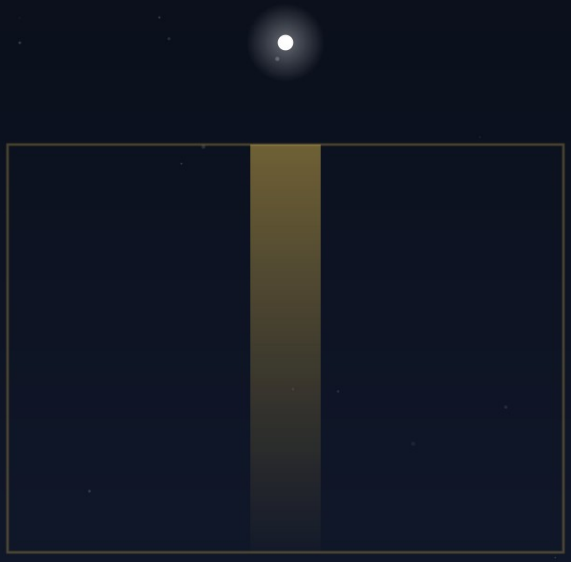

THE BREATH OF STONE, VOLUME III

The Unconstructible Seven

One clause confirmed, one refuted

TRILOGY II OF STONE

KEITH SHIRLEY

The Breath of Stone, Volume III

The Unconstructible Seven

A Conclave Study in Two Hands

Keith Shirley

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What was predicted, and what came back

P5, sealed before searching:

> Regular seven-fold geometry is **rare to absent** in traditions working by compass-and-cord or straightedge methods, and where seven-fold division occurs it is **approximate** or achieved by counting rather than construction.
>
> Pentagons are the control. Five-fold geometry *is* constructible, so if seven is rare because of geometric difficulty rather than symbolic avoidance, five should be noticeably commoner than seven.

The prediction has **two clauses and they came back differently.**

Approximate where present: confirmed, and sharply.

Rare to absent: not supported. Seven-fold work exists, has a historical corpus, and a practitioner's account calls it uncommon but not rare [531].

And the failed clause is the more interesting one, because it means the craftsmen did not avoid the impossible number.

They built it anyway, approximately, and it worked.

THE THIRTEEN BREATHS

Breath 1. The Fact

Which is not symbolic and not arguable

A regular heptagon cannot be constructed with compass and unmarked straightedge. This is not a historical observation. It is a theorem.

Seven is a Pierpont prime but not a Fermat prime, and constructible regular polygons are those whose side count is a power of two multiplied by distinct Fermat primes. The heptagon is **constructible with a marked ruler and compass, and it is the smallest regular polygon with that property** [517, 525].

Gauss characterised which regular polygons are constructible, announcing the seventeen-sided case on 30 March 1796 before getting out of bed, at nineteen [522]. He and Wantzel between them proved the other half: that the heptagon and the hendecagon are impossible [520].

So the sequence of small regular polygons sorts into two classes, and the sort is absolute. **Three, four, five, six, eight, ten, twelve, fifteen, sixteen, seventeen: constructible. Seven, nine, eleven, thirteen, fourteen: not.**

Seven is the first failure.

Breath 2. The Gap in Euclid

The direct historical evidence, and it is a silence

Euclid's *Elements*, Book IV, inscribes regular polygons in a circle: the triangle, the square, the pentagon, the hexagon and the fifteen-gon [524].

Not the heptagon.

And the fifteen is the tell. Fifteen is three times five, and Euclid reaches it by combining the triangle and the pentagon he has already built. He is constructing composites from his available primes.

Seven is prime and unavailable, so it does not appear, and Euclid does not explain its absence because he cannot. The proof that it is impossible is two thousand and ninety-six years away.

This is the strongest evidence in the volume and it is worth being precise about what it shows. **It does not show that the Greeks knew the heptagon was impossible.** It shows that the foundational text of Greek geometry contains a gap exactly where the mathematics says a gap must be, and that nobody filled it for two millennia of sustained effort.

That is what a genuine barrier looks like in a record. Not a prohibition. An absence that holds under pressure.

Grade: established. The Euclidean contents are not in dispute; the reading of the gap is the previous trilogy's method applied to a mathematical canon.

Breath 3. Archimedes Bending the Rules

The exception that measures the barrier

A heptagon construction is attributed to Archimedes, and it works. It uses *neusis*, a marked ruler, sliding a measured segment until it fits.

That is precisely the tool the theorem says is required: seven is the smallest regular polygon needing a marked ruler [517].

And a practitioner's account of Islamic geometric drawing records the crucial detail: the exact Greek construction exists, and **it is so obscure that it is never used** [531].

Hold that beside Breath 2. The construction was found. It was correct. And it did not enter practice, in two thousand years, anywhere.

Because a construction that nobody can remember or execute reliably is, for working purposes, not a construction.

The barrier was not that seven could not be drawn. It was that seven could not be drawn the way everything else was drawn.

Breath 4. The Control

Five, which is constructible and is everywhere

The pentagon control was registered because five-fold geometry **is** constructible and is in Euclid.

In Islamic geometric ornament, the canonical system is built on five and ten. The *girih* tile set comprises five tiles: a regular decagon, an elongated hexagon, a bow tie, a regular pentagon and a

rhombus, all of equal side length and **all of their angles multiples of 36 degrees**, which is a tenth of a circle [529, 526].

That is a complete, standardised, transmissible design system resting on the fivefold division. It has a name, a tile set, and a literature.

There is no equivalent sevenfold system in the historical record. Modern researchers have had to construct one: Bonner's Bridges paper presents a comprehensive set of sevenfold sub-grid elements as a contribution, noting that while a number of historical sevenfold examples exist, the historical record gives insufficient evidence to establish how far designers used systematic methods [528]. A more recent paper presents a method to generate sevenfold girihs across all systems and families, again as new work [530].

The control holds. Five is a system. Seven is a specialist reconstruction. The asymmetry is exactly where constructibility predicts.

Breath 5. The Tessellation Confound, and Why It Fails

The alternative explanation, and why it does not survive the control

There is a rival account of heptagon rarity that has nothing to do with construction: regular heptagons do not tile the plane [521].

That is true. It is also true of regular pentagons. Only the triangle, the square and the hexagon tile the plane regularly.

So if non-tiling explained the rarity of seven, it would explain the rarity of five equally.

It does not. Five is the backbone of the single most elaborate ornamental tradition in the history of geometry, achieved by using pentagons and decagons as sub-grid elements in non-regular tilings [526, 529].

The craftsmen demonstrably had the technique for building tilings out of non-tiling regular polygons. They applied it to five and not to seven.

The confound is neutralised by the control, which is what the control was registered for.

Grade: established as reasoning.

Breath 6. Uncommon But Not Rare

Where the prediction failed

P5 said rare to absent. The record says otherwise.

Seven- and fourteen-arm stars and seven-sided figures are **uncommon but not rare** in Islamic geometric art [531]. Historical sevenfold patterns exist in sufficient number to form a corpus [528, 530]. A practitioner's guide to dividing the circle presents constructions for three, four, five, six, **seven**, eight, nine, ten, twelve, fourteen, sixteen and twenty-fold division as part of the working repertoire [531].

So seven was in the toolkit. Craftsmen divided circles into sevenths and built from them.

The prediction was wrong, and it was wrong because it assumed that mathematical impossibility would act as a practical prohibition.

It did not. And why it did not is the volume's finding.

Breath 7. Pseudoheptagonal

The clause that confirmed, and the word that does it

In a crystallographic analysis of Moroccan ornamental patterns, the authors describe a design in which **the eightfold elements are regular whereas the sevenfold ones are pseudoheptagonal approximations** [527].

That is P5's second clause, stated by specialists analysing an actual building, in a single sentence, with a technical term for it.

Within one pattern, on one wall, the constructible symmetry is exact and the non-constructible one is approximate. The craftsman did both, on the same surface, and the mathematics shows through in the difference.

Grade: established as to that pattern; **supported** as a general claim, because this study examined one analysed example and not a survey.

Breath 8. The Tolerance

Why the barrier was ignored, and this is the answer

Put Breath 6 and Breath 7 together and the resolution is immediate.

Seven is impossible **exactly**. Construction is a question about ideal points and infinite precision.

A building is not made of ideal points. A heptagonal star in tile work has sides of some real length cut by a person with a blade, set in mortar, on a wall that is not perfectly flat.

An approximate heptagon, at architectural scale, is visually indistinguishable from an exact one. The best approximations are accurate to small fractions of a per cent, and no eye, and no chisel, resolves that.

So the theorem's barrier is **real in mathematics and absent in craft**, because craft operates inside a tolerance that mathematics does not have.

And that is the previous trilogy's finding arriving in a new domain. **Coarseness is protection.** There, a star compass with 11.25-degree houses absorbed a millennium of precessional drift. Here, a tolerance of a tenth of a millimetre absorbs an impossibility proved by Gauss.

The unconstructible seven was constructed anyway, slightly wrongly, and it has been holding up walls for a thousand years.

Grade: supported, and it is the volume's central finding.

Breath 9. What the Difficulty Actually Cost

The honest residue of P5

If approximation works, why is seven uncommon at all?

Not prohibition, and the record contains no evidence of avoidance. Three ordinary costs, each small and cumulative.

It takes longer. Every sevenfold layout requires an approximation procedure rather than a remembered construction.

It does not transmit as a system. Five has a tile set with named elements that a workshop can teach. Seven has no such canonical system in the historical record, which is why modern researchers are still building one [528, 530].

And it compounds badly. An approximation repeated across a large field accumulates error, where an exact construction does not.

So seven is not forbidden. **It is expensive**, in time, in teachability and in scale, and expensive things are done less often.

That is a duller explanation than symbolic avoidance and it fits the evidence better.

Breath 10. The Temptation

Recorded because the sealed file warned about it

The file flagged, as suggestive only, that seven is both the first number a builder cannot construct and a number sitting past the limit at which an observer can enumerate a compact group.

Having now worked the material, **the resemblance should be discounted.**

They are unrelated mechanisms. The enumeration limit is a fact about parallel attention capacity, around four, and the Pleiades falls past it for reasons of angular separation. The construction limit is a fact about field extensions and Fermat primes. Seven lands past both by coincidence.

And the two behave oppositely, which is the decisive point. **The enumeration barrier cannot be approximated around.** Nobody ever looked harder and got the seventh Pleiad. **The construction barrier can be, and routinely was.**

Had this volume been written to confirm the project's existing findings, that resemblance is exactly the kind of thing that would have been promoted into an illumination.

It is recorded here as declined.

Breath 11. The Stone

What this means for Peru, which is thin

The field protocol asks for a count of sides on any polygonal plan, enclosure, niche arrangement or regular division of a circle, with five-fold as the control.

This volume supplies the prior: **expect few sevens, and expect any that appear to be approximate rather than regular.** Expect the approximation to be invisible without measurement, per Breath 8.

But Inca construction is not a compass-and-straightedge tradition in the Greek or Islamic sense. The masonry is polygonal and irregular by design, each stone cut to its neighbours with no

module, as Volume I established. A tradition that does not standardise is a tradition for which the constructibility question may simply not arise.

So the Peruvian application of P5 is **weak by construction**, and this is stated now rather than after the count comes back ambiguous. The strong evidence for P5 is Mediterranean and Islamic, not Andean.

Breath 12. The Two-Thousand-Year Silence

What the history actually looks like

Set the dates out and the shape is unmistakable.

Around 300 BCE, Euclid inscribes 3, 4, 5, 6 and 15, and omits 7 [524].

Around 250 BCE, Archimedes finds a heptagon by bending the rules, and the construction sinks without trace into obscurity [524, 531].

For roughly two thousand years, approximations proliferate [518, 519, 521, 523, 525, 532], craftsmen build pseudoheptagons, and nobody can say why the exact version will not come.

30 March 1796, a nineteen-year-old works out which polygons are constructible and why [522].

And shortly after, Wantzel proves the impossibility, closing a question Euclid had left open by leaving it out [520].

Two millennia of sustained effort by competent people, producing approximations and no proof of their necessity.

That is what a real barrier looks like in a historical record. Not a taboo, not a secret, not a suppressed knowledge: **a persistent, unexplained practical failure, with workarounds, and an eventual proof that the failure was necessary.**

Worth comparing to the other great unexplained practical failure in this project. Nobody could count the Pleiades either. Fourteen traditions worked around it with a collective noun, and the explanation arrived in a psychology laboratory in 1949.

Breath 13. The Return

What this volume is for

It is for one correction to the trilogy's own instincts.

The project has repeatedly found that limits in the world shape what people make and say: an enumeration threshold produces a word for *heap*, a precessional drift produces a fossilised canon, an aperture width bounds an alignment.

The heptagon looked like another of those, and it is not.

Here the limit was real, absolute, provable, and comprehensively ignored. People wanted seven-sided things and made them, a fraction of a per cent out of true, and the fraction never mattered to anybody.

Which is a useful thing for a project about thresholds to have found. **Not every barrier in the world is a barrier in practice.** The ones that bite are the ones you cannot approximate your way around.

The Pleiades is one of those. The heptagon is not.

WHAT WAS SEEN

- Seven is the smallest regular polygon requiring a marked ruler, and the first non-constructible one.
- Euclid Book IV builds 3, 4, 5, 6 and 15, and omits 7, with the 15 showing he was composing from available primes.
- An exact Archimedean construction exists and was never adopted in practice.
- Five is a complete transmissible system with a named tile set; seven has no historical equivalent and is being built by modern researchers.
- In an analysed Moroccan pattern, eightfold elements are regular and sevenfold ones are explicitly pseudoheptagonal.
- Seven-fold work is uncommon but not rare, and seven-fold circle division is in the practitioner's repertoire.

WHAT WAS UNSEEN

- Any evidence of symbolic avoidance of seven in ornament.
- Any canonical historical sevenfold design system.
- Any sign that approximation troubled anyone.
- Any architectural census of heptagonal plans, which this volume did not perform and which P5's first clause really required.

THE GEOMETRY BENEATH

128.571 degrees. The interior angle of a regular heptagon, which is 900 divided by 7, and does not resolve.

36 degrees. A tenth of a circle, and the unit of the entire girih system.

A fraction of one per cent. The error in a good heptagon approximation, and the reason the theorem never mattered to a craftsman.

Two thousand and ninety-six years. Euclid to Gauss.

THE LEDGER

Claim	Grade
The regular heptagon is not constructible with compass and unmarked straightedge	established, as theorem
It is the smallest regular polygon requiring a marked ruler	established
Euclid Book IV omits the heptagon while building 3, 4, 5, 6 and 15	established
That omission reflects the mathematical barrier	supported
Greeks knew the heptagon was impossible	**not supported**; no proof existed until the 19th century
An Archimedean neusis construction exists and was never used in practice	supported
Five-fold has a canonical transmissible system; seven-fold does not	supported
The non-tessellation explanation is neutralised by the pentagon control	established, as reasoning
P5, clause one: seven-fold is rare to absent	**not supported**; uncommon but not rare
P5, clause two: approximate where present	**established** at the analysed Moroccan case; supported generally
Seven is expensive rather than forbidden	supported
The resemblance between the enumeration and construction limits	**declined**; unrelated mechanisms, opposite behaviour
P5 applied to Inca construction	weak by construction; stated in advance

NEAR-TRUTHS

- **“The ancients avoided the heptagon because seven was sacred.”** They did not avoid it. They built it, approximately, in tile and in stars, and nothing in the record suggests reluctance. What they lacked was a construction, and what they used instead was good enough.
- **“The Greeks knew the heptagon was impossible.”** They knew they could not do it. Those are different, and the gap between them lasted twenty-one centuries. Knowing a thing cannot be done is a mathematical achievement in its own right and it belongs to Gauss and Wantzel.
- **“Impossible means it does not appear.”** The most reliable lesson in this volume. An exact impossibility and a practical impossibility are different claims, and only one of them shows up in the built world.

THREE ILLUMINATIONS

The First: The Impossible Thing Was Built Anyway

Gauss proved it cannot be done. Wantzel closed the proof. The theorem is absolute and admits no exception.

And there are seven-pointed stars on walls in Morocco, and seven-sided figures in the repertoire of working geometers, and a practitioner today teaches the sevenfold division of a circle alongside the fivefold and the eightfold [531].

The craftsmen were not defying mathematics and were not ignorant of the difficulty. They were **operating in a different unit**. The theorem is about ideal points; the wall is about tile edges and mortar and a blade.

A heptagon accurate to a fraction of a per cent is, on a wall, a heptagon. There is no instrument in the building trade that disagrees.

So the barrier is real where it lives and absent where people work, and the distance between those two places is about a tenth of a millimetre.

Grade: supported.

The Second: The Pentagon Control Did Its Job, and That Is Why This Is Worth Anything

P5 would have been unfalsifiable without it.

Seven-fold geometry is uncommon. That alone proves nothing, because almost everything is uncommon, and a hundred explanations are available: taste, symbolism, tessellation, accident.

The control removes them all at once. **Five is also non-tiling. Five is also odd. Five is also prime.** Five differs from seven in exactly one respect that matters: it can be constructed.

And five is the backbone of an entire ornamental civilisation while seven is a specialist curiosity that modern scholars are still systematising.

One variable, one difference in outcome. That is the whole method of the previous trilogy, applied to a question about geometry rather than about the sky, and it works the same way.

Without the control, this volume would have been a pattern noticed and dressed as a finding.

Grade: established, as reasoning.

The Third: Only Some Limits Bite

This project has been a catalogue of thresholds that shape what people build and say. The enumeration limit at four produces a word for *heap* in fourteen languages. Precession produces fossilised canons in three literate traditions. Aperture width bounds what any alignment can mean.

The heptagon was meant to be the fourth, and it is not, and the reason is the most useful thing here.

Compare the two barriers directly.

Nobody ever resolved the seventh Pleiad by trying harder. The enumeration limit cannot be approximated around; there is no partial credit, no version of looking that gets you most of the way to a count that is not available. Every culture that wanted seven had to go outside observation to get it, and then build a story to explain the gap.

The construction limit is absolute in the same formal sense and **completely permeable in practice**. You can get to within a fraction of a per cent with a procedure anybody can learn, and the remaining error is smaller than your chisel.

So the thresholds that shape human culture are not simply the true ones. **They are the ones with no approximation available.**

That is why there are lost-sister stories on five continents and no lost-heptagon stories anywhere. One limit left a hole that had to be filled with a narrative. The other left a hole you could fill with mortar.

Grade: supported, and it is the correction this volume contributes to the trilogy that produced it.

> ✦ **Practice: the barrier you can walk around.** Take a circle, drawn by any means, and a way of marking it. A plate and a pencil will do. Breathe gently, in through the nose, out a little longer, four or five times. Divide it into six. This is easy and exact, and your hand will find it: the radius steps round the circumference six times and closes

perfectly. Now divide the same circle into seven. Try it. Step the compass round and adjust and step again. You will get there. It will take several passes and it will close, and the closure will be a little forced. **Nothing you can see will be wrong with it.** And it is provably not a heptagon, and no amount of care will make it one. Hold both facts for one breath: that it is exactly impossible, and that you just did it. Most of what gets built in the world sits in that gap. Breathe gently, and never force or hold the breath.

WHAT THE TRILOGY HAS AFTER THREE

Volume I is unscored and honest about it. The aperture measurement and the golden-section test need a person at five walls with a tape. Everything else in it is scaffolding for those numbers.

Volume II is established. Restoration inflates apparent precision. The canon-demotion mechanism inferred from four-thousand-year-old tablets is visible in a 1960s guidebook decision. And a false canon in stone can become load-bearing, which has no counterpart in the previous trilogy.

Volume III is half confirmed and half refuted, and the refutation is the better half. Seven-fold geometry is not rare, it is approximate; the barrier is real in mathematics and permeable in craft; and only limits with no available approximation leave marks on culture.

And the trilogy as a whole has one correction to make to the trilogy before it. *The Edge of Counting* proposed that the thresholds people share are perceptual. That stands. What this trilogy adds is that **not every real limit becomes a cultural one.** The test is whether you can get close enough and move on.

The Pleiades could not be approximated. The heptagon could. One of them is in fourteen languages and the other is a footnote in Euclid.

A note on the mirror

The mathematics here is secure and is the strongest material in the trilogy; it was read through encyclopaedia and expository sources rather than through Gauss or Wantzel directly, but the content is standard and uncontested.

The architectural evidence is thinner than the prediction required. **P5's first clause asked about rarity and this study performed no census.** The finding that seven-fold is uncommon but not rare rests substantially on one practitioner's account, and the pseudoheptagonal finding rests on one analysed Moroccan pattern. A proper test would code a corpus of Islamic geometric patterns by symmetry order, which is a feasible project and has not been done here.

The Euclid reading was taken from a secondary sacred-geometry source; the contents of Book IV are standard and verifiable in any edition, and a reader should verify rather than take it on this study's word.

No Arabic, Persian or Greek source was read. This is a mirror's work, held up for the author to check.

Sources

517. Heptagon, Wikipedia, on Pierpont primes and the marked-ruler construction. https://en.wikipedia.org/wiki/Heptagon
518. The Mysterious Heptagon, on the Galois-theoretic result. https://epko.gov.gr/arts/the-mysterious-heptagon-unveiling-the-secrets-of-the-seven-sided-shape
519. The Heptagon, on the existence of many approximations. https://www.academia.edu/11553041/The_Heptagon
520. Why This Great Mathematician Wanted a Heptadecagon on His Tombstone, Scientific American, on Gauss and Wantzel proving the heptagon and hendecagon impossible. https://www.scientificamerican.com/article/why-this-great-mathematician-wanted-a-heptadecagon-on-his-tombstone/
521. Heptagon Objects, on non-tessellation. https://brainbound.blog/heptagon-objects-real-world-examples
522. The Mighty Heptadecagon, on Gauss on 30 March 1796. https://www.arondale.com/heptadecagon
523. Mastering Heptagons, on the impossibility proven in the 19th century. https://www.storyofmathematics.com/heptagon/
524. Sacred Geometry 7: The Heptagon and the Seven Circles, on Euclid Book IV and Archimedes bending the rules. Secondary source; Book IV contents are verifiable in any edition. https://templeofzeus.org/sacred-geometry-heptagon-and-seven-circles
525. Heptagon, Design Encyclopedia, on Fermat primes. https://design-encyclopedia.com/?T=Heptagon
526. Girih tile, Wikipedia, on the fivefold and tenfold symmetries of the tile set. https://en.wikipedia.org/wiki/Girih_tile
527. Moroccan ornamental quasiperiodic patterns constructed by the multigrid method, on regular eightfold elements and pseudoheptagonal approximations. https://www.researchgate.net/figure/A-tenfold-pattern-Madrasa-Ben-Youssef-Marrakech_fig9_261376894
528. J. Bonner, A 7-Fold System for Creating Islamic Geometric Patterns, Bridges 2012. https://archive.bridgesmathart.org/2012/bridges2012-141.pdf

529. Methods and apparatus for creating girih strapwork patterns, on the five tiles and their 36-degree angles.
https://image-ppubs.uspto.gov/dirsearch-public/print/downloadPdf/9936597
530. Presenting a Method to Generate Existing and Novel Girihs for All Systems and Families Based on 7-Fold Patterns, Computer-Aided Design.
https://www.sciencedirect.com/science/article/abs/pii/S0010448522001518
531. Divided Circles: The Foundations of Geometric Pattern, Drawing Islamic Geometric Designs, on the obscurity of the Archimedean construction and sevenfold figures being uncommon but not rare.
https://www.drawingislamicgeometricdesigns.com/blog/divided-circles-the-foundations-of-geometric-pattern
532. Overlapping circles grid, Wikipedia, on straightedge-and-compass girih construction.
https://en.wikipedia.org/wiki/Overlapping_circles_grid

In Harmony, Small Things Grow Towards Unity.